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# Auxiliary "massless" spin-2 field in de Sitter universe

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**Abstract**

For the tensor field of rank-2 there are two unitary irreducible representation (UIR) in de Sitter (dS) space denoted by $\Pi^{\pm}_{2,2}$ and $\Pi^{\pm}_{2,1}$ [1]. In the flat limit only the $\Pi^{\pm}_{2,2}$ coincides to the UIR of Poincaré group, the second one becomes important in the study of conformal gravity. In the pervious work, Dirac's six-cone formalism has been utilized to obtain conformally invariant (CI) field equation for the "massless" spin-2 field in dS space [2]. This equation results in a field which transformed according to $\Pi^{\pm}_{2,1}$, we name this field the auxiliary field. In this paper this auxiliary field is considered and also related two-point function is calculated as a product of a polarization tensor and "massless" conformally coupled scalar field. This two-point function is de Sitter invariant.

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# 1 Introduction

Recent astrophysical data indicate that our universe might currently be in a dS phase. Quantum field theory in dS space-time has evolved as an exceedingly important subject, studied by many authors in the course of past decade. The importance of dS space has been primarily ignited by the study of the inflationary model of the universe and quantum gravity. The importance of the massless spin-2 field in the dS space is due to the fact that it plays the central role in quantum gravity and quantum cosmology in the linear approximation. In dS space, mass is not an invariant parameter for the set of observable transformations under the dS group $SO(1,4)$. Concept of light-cone propagation, however, does exist and leads to the conformal invariance. "Massless" is used in reference to propagation on the dS light-cone (conformal invariance). The term "massive" is refereed to fields that in their Minkowskian limit (zero curvature) reduce to massive Minkowskian fields [3].

It has been shown that the massive and massless conformally coupled scalar fields in dS space correspond to the principal and complementary series representations of dS group, respectively [4]. The massive vector field in dS space has been associated with the principal series, whereas massless field corresponds to the lowest representation of the vector discrete series representation in dS group [5]. The massive and massless spin-2 fields in dS space have been also associated with the principal series and the lowest representation of the rank-2 tensor discrete series of dS group, respectively [6, 7, 8].

It has been shown that CI wave equation for the tensor field of rank two must obey following fourth order derivative equation [2]

$$(Q_2+6)(Q_2+4)\mathcal{K}_{\alpha\beta}=0,$$

where $Q_2$ is the Casimir operator of the dS group. This equation can be considered as two equation of order 2

$$(Q_2+6)\mathcal{K}_{\alpha\beta}=0, \quad \text{transforms according to } \Pi^{\pm}_{2,2},$$

$$(Q_2+4)\mathcal{K}_{\alpha\beta}=0, \quad \text{transforms according to } \Pi^{\pm}_{2,1}.$$

In the flat limit ($H\rightarrow 0$) only $\Pi^{\pm}_{2,2}$ coincides to the UIR of the Poincaré group, however both representations become important in studding the CI field equation in dS space. The representation $\Pi^{\pm}_{2,2}$ has been studied previously, in this paper we study the fields which transform according to $\Pi^{\pm}_{2,1}$. Since there is no flat limit to these fields so they are not detectable locally, and such fields named as auxiliary fields.

The organization of this paper and its brief outlook are as follows. Section 2 is devoted to a brief review of the dS massless spin-2 field equations in the ambient space and aslo Dirac's manifestly covariant formalism has been reviewed. In Section 3 we study the auxiliary CI field equations and their solutions. We show that this field can be written in terms of a polarization tensor and a massless conformally coupled scalar field as follows

$$\mathcal{K}_{\alpha\beta}(x)=\mathcal{D}_{\alpha\beta}(x,\partial)\phi(x).$$

In section 4 we calculate two-point function $\mathcal{W}_{\alpha\beta\alpha'\beta'}(x,x')$ in 5 dimensional flat space (ambient space notation) for the field equation. Finally a brief conclusion and an outlook for further investigation has been presented. We have supplied some useful mathematical details of calculations in the appendices.

# 2 de Sitter space and Dirac's six-cone formalism

***I) de Sitter space:*** The dS space-time can be defined by the one-sheeted four-dimensional hyperboloid:

$$X_H = \{x \in \mathbb{R}^5; x^2 = \eta_{\alpha\beta}x^\alpha x^\beta = -H^{-2} = -\frac{3}{\Lambda}\}, \quad \alpha, \beta = 0, 1, 2, 3, 4, \tag{2.1}$$

where $\eta_{\alpha\beta} = \text{diag}(1, -1, -1, -1, -1)$ and $H$, $\Lambda$ are the Hobble parameter and cosmological constant respectively. The dS metric is

$$ds^2 = \eta_{\alpha\beta}dx^\alpha dx^\beta = g^{dS}_{\mu\nu}dX^\mu dX^\nu, \quad \mu, \nu = 0, 1, 2, 3,$$

where the $X^\mu$'s are 4 space-time intrinsic coordinates of the dS hyperboloid. Any geometrical object in this space can be written in terms of the four local coordinates $X^\mu$ (intrinsics) or in terms of the five global coordinates $x^\alpha$ (ambient space).
The wave equation for massless tensor fields $h_{\mu\nu}(X)$, propagating in de Sitter space is [9, 10]:

$$(\Box_H + 2H^2)h_{\mu\nu} - (\Box_H - H^2)g^{dS}_{\mu\nu}h' - \mathcal{S}\nabla_\mu\nabla \cdot h_\nu$$
$$+g^{dS}_{\mu\nu}\nabla \cdot \nabla \cdot h + \nabla_\mu\nabla_\nu h' = 0, \tag{2.2}$$

where $\Box_H$ is the Laplace-Beltrami operator in dS space, $h' = h^\mu_\mu$ and $\mathcal{S}(\alpha\beta) = (\alpha\beta + \beta\alpha)$. $\nabla^\nu$ is the covariant derivative in dS space. The field equation (2.2) is invariant under the following gauge transformation

$$h_{\mu\nu} \longrightarrow h^{gt}_{\mu\nu} = h_{\mu\nu} + \mathcal{S}\nabla_\mu\chi_\nu, \tag{2.3}$$

where $\chi_\nu$ is an arbitrary vector field. A general family of gauge conditions can be chosen

$$\nabla^\mu h_{\mu\nu} = \zeta\nabla_\nu h', \tag{2.4}$$

where $\zeta$ is an arbitrary constant. The choice of $\zeta = \frac{1}{2}$ makes the relation between field and group representation thoroughly apparent in the ambient space [11]. In the following, ambient space notations is used; in ambient space, the relationship with UIRs of the dS group becomes straightforward because the Casimir operators are easy to identify. The tensor field $\mathcal{K}_{\alpha\beta}$ defined by $\mathbb{R}^5$-variables $x^\alpha$ in de Sitter space-time. Useful relations between ambient and intrinsic coordinates are listed in the appendix. The kinematical group of the dS space is the 10-parameter group $SO_0(1, 4)$ (connected component of the identity in $SO(1, 4)$ ), which is one of the two possible deformations of the Poincaré group. There are two Casimir operators

$$Q^{(1)}_2 = -\frac{1}{2}L^{\alpha\beta}L_{\alpha\beta}, \qquad Q^{(2)}_2 = -W_\alpha W^\alpha, \tag{2.5}$$

where

$$W_\alpha = -\frac{1}{8}\epsilon_{\alpha\beta\gamma\sigma\eta}L^{\beta\gamma}L^{\sigma\eta}, \tag{2.6}$$

with 10 infinitesimal generators $L_{\alpha\beta} = M_{\alpha\beta} + S_{\alpha\beta}$. The subscript 2 in $Q^{(1)}_2$, $Q^{(2)}_2$ reminds that the carrier space is constituted by second rank tensors. The orbital part $M_{\alpha\beta}$, and the action of the spinorial part $S_{\alpha\beta}$ on a tensor field $\mathcal{K}$ defined on the ambient space read respectively [12]

$$M_{\alpha\beta} = -i(x_\alpha\partial_\beta - x_\beta\partial_\alpha) = -i(x_\alpha\bar{\partial}_\beta - x_\beta\bar{\partial}_\alpha),$$

$$S_{\alpha\beta}\mathcal{K}_{\gamma\delta} = -i(\eta_{\alpha\gamma}\mathcal{K}_{\beta\delta} - \eta_{\beta\gamma}\mathcal{K}_{\alpha\delta} + \eta_{\alpha\delta}\mathcal{K}_{\beta\gamma} - \eta_{\beta\delta}\mathcal{K}_{\alpha\gamma}). \tag{2.7}$$

The symbol $\epsilon_{\alpha\beta\gamma\sigma\eta}$ holds for the usual antisymmetric tensor. $\bar{\partial}_\alpha$ is the tangential (or transverse) derivative on dS space defined by

$$\bar{\partial}_\alpha = \theta_{\alpha\beta}\partial^\beta = \partial_\alpha + H^2 x_\alpha x\cdot\partial, \;\text{ with, }\; x\cdot\bar{\partial} = 0\,,$$

and $\theta_{\alpha\beta}$ is the transverse projector ($\theta_{\alpha\beta} = \eta_{\alpha\beta} + H^2 x_\alpha x_\beta$ ).

The operator $Q_2^{(1)}$ commutes with the action of the group generators and, as a consequence, it is constant in each UIR. Thus the eigenvalues of $Q_2^{(1)}$ can be used to classify UIRs *i.e.,*

$$(Q_2^{(1)} - \langle Q_2^{(1)}\rangle)\mathcal{K}(x) = 0. \tag{2.8}$$

Following Dixmier [1], we get a classification scheme using a pair $(p, q)$ of parameters involved in the following possible spectral values of the Casimir operators :

$$Q_p^{(1)} = (-p(p+1) - (q+1)(q-2))\, I_d, \qquad Q_p^{(2)} = (-p(p+1)q(q-1))\, I_d\,. \tag{2.9}$$

Three types of scalar, tensorial or spinorial UIRs are distinguished for $SO(1,4)$ according to the range of values of the parameters $q$ and $p$ [1, 13], namely: the principal, the complementary and the discrete series. The flat limit indicates that for the principal and the complementary series the value of $p$ bears meaning of spin. For the discrete series case, the only representation which has a physically meaningful Minkowskian counterpart is $p = q$ case. Mathematical details of the group contraction and the physical principles underlying the relationship between dS and Poincaré groups can be found in Ref.s [14] and [15] respectively. The spin-2 tensor representations relevant to the present work are as follows:

i) The UIRs $U^{2,\nu}$ in the principal series where $p = s = 2$ and $q = \frac{1}{2} + i\nu$ correspond to the Casimir spectral values:

$$\langle Q_2^\nu\rangle = \nu^2 - \frac{15}{4}, \quad \nu \in \mathbb{R}, \tag{2.10}$$

note that $U^{2,\nu}$ and $U^{2,-\nu}$ are equivalent.

ii) The UIRs $V^{2,q}$ in the complementary series where $p = s = 2$ and $q - q^2 = \mu$, correspond to

$$\langle Q_2^\mu\rangle = q - q^2 - 4 \equiv \mu - 4, \quad 0 < \mu < \frac{1}{4}\,. \tag{2.11}$$

iii) The UIRs $\Pi^{\pm}_{2,q}$ in the discrete series where $p = s = 2$ correspond to

$$\langle Q_2^{(1)}\rangle = -4, \;\; q = 1\; (\Pi^{\pm}_{2,1}); \;\; \langle Q_2^{(2)}\rangle = -6, \;\; q = 2\; (\Pi^{\pm}_{2,2}), \tag{2.12}$$

the sign $\pm$ stands for the helicity of the representation.

The action of the Casimir operators $Q_1$ and $Q_2$ can be written in the more explicit form

$$Q_1 K_\alpha = (Q_0 - 2)K_\alpha + 2x_\alpha \partial.K - 2\partial_\alpha x\cdot K, \tag{2.13}$$

$$Q_2\mathcal{K}_{\alpha\beta} = (Q_0 - 6)\mathcal{K}_{\alpha\beta} + 2\mathcal{S}x_\alpha\partial\cdot\mathcal{K}_\beta - 2\mathcal{S}\partial_\alpha x\cdot\mathcal{K}_\beta + 2\eta_{\alpha\beta}\mathcal{K}', \tag{2.14}$$

where $\mathcal{K}'$ is the trace of $\mathcal{K}_{\alpha\beta}$ and $Q_0 = \frac{1}{2}M_{\alpha\beta}M^{\alpha\beta}$.

***II) Dirac's six-cone formalism:*** Dirac's six-cone formalism originally was defined by Dirac to obtain CI wave equation [16]. This formalism developed by Mack and Salam [17] and many others. Dirac's six-cone or Dirac's projection cone is defined by

$$u^2 \equiv u_0^2 - \vec{u}^2 + u_5^2 = \eta^{ab}u_a u_b = 0, \quad \eta^{ab} = diag(1,-1,-1,-1,-1,1), \tag{2.15}$$

where $u_a \in \mathbb{R}^6$, and $\vec{u} \equiv (u_1, u_2, u_3, u_4)$. Reduction to four dimensional (physical space-time) is achieved by projection, that is by fixing the degrees of homogeneity of all fields. Wave equations, subsidiary conditions, etc., must be expressed in terms of operators that are defined intrinsically on the cone. These are well-defined operators that map tensor fields on tensor fields with the same rank on cone $u^2 = 0$ [18, 10]. It is important to note that on the cone $u^2 = 0$, the second order Casimir operator of conformal group, $\mathcal{Q}_2$, is not a suitable operator to obtain CI wave equations. Because for a symmetric tensor field of rank-2, we have [3, 10, 19]:

$$\mathcal{Q}_2\Psi^{cd} = \frac{1}{2}L_{ab}L^{ab}\Psi^{cd} = \left(-u^2\partial^2 + \hat{N}_5(\hat{N}_5+4)+8\right)\Psi^{cd},$$

where $\hat{N}_5$ is the conformal-degree operator defined by $\hat{N}_5 \equiv u^a\partial_a$.
On the cone this operator reduces to a constant, *i.e.* $\hat{N}_5(\hat{N}_5+4)+8$. It is clear that this operator cannot lead to the wave equations on the cone. The well-defined operators exist only in exceptional cases. For tensor fields of degree $-1, 0, 1, ...$, the intrinsic wave operators are $\partial^2, (\partial^2)^2, (\partial^2)^3, ...$ respectively [10]. Thus the following CI system of equations, on the cone, has been used [18]:

$$\left\{ \begin{array}{rcl} (\partial_a\partial^a)^n\Psi &=& 0, \\ \hat{N}_5\Psi &=& (n-2)\Psi. \end{array} \right. \tag{2.16}$$

where $\Psi$ is a tensor field of a definite rank and of a definite symmetry.

Other CI conditions can be added to the above system in order to restrict the space of the solutions. The conditions *i*- transversality: $u_a\Psi^{ab...} = 0$, *ii*- tracelessness: $\Psi^a_{ab...} = 0$ and *iii*- divergencelessness: $Grad_a\Psi^{ab...} = 0$ are introduced to achieve the above goal.
The operator $Grad_a$ unlike $\partial_a$ is intrinsic on the cone, and is defined by [10]:

$$Grad_a \equiv u_a\partial_b\partial^b - (2\hat{N}_5+4)\partial_a\,. \tag{2.17}$$

In order to project the coordinates on the cone $u^2 = 0$, to the $1+4$ dS space we chose the following relation:

$$\left\{ \begin{array}{rcl} x^\alpha &=& (Hu^5)^{-1}u^\alpha, \\ x^5 &=& Hu^5. \end{array} \right. \tag{2.18}$$

Note that $x^5$ becomes superfluous when we deal with the projective cone. It is easy to show that various intrinsic operators introduced previously now read as:

1. the conformal-degree operator ($\hat{N}_5$)

$$\hat{N}_5 = x_5\frac{\partial}{\partial x_5}, \tag{2.19}$$

2. the conformal gradient ($Grad_\alpha$)

$$Grad_\alpha = -x_5^{-1}\{H^2 x_\alpha[Q_0 - \hat{N}_5(\hat{N}_5 - 1)] + 2\bar{\partial}_\alpha(\hat{N}_5 + 1)\}, \tag{2.20}$$

3. and the powers of d'Alembertian $(\partial_a\partial^a)^n$, which acts intrinsically on field of conformal degree $(n-2)$,

$$(\partial_a\partial^a)^n = -H^{2n}x_5^{-2n}\prod_{j=1}^{n}[Q_0 + (j+1)(j-2)]\,. \tag{2.21}$$

In the next section we consider CI field equation with $n = 1$ in (2.16) and its possible solution.

# 3 Auxiliary CI equation and its solution

Dirac's six-cone formalism provides us with the opportunity to obtain CI wave equations for scalar, vector and tensor fields. It is shown [18] that for the scalar and vector field, the simplest CI system of equations is obtained through $n = 1$ in (2.16), *i.e.* the field with conformal degree $-1$. Resulting equations are the UIRs of $SO(1,4)$. In the flat limit ($H \to 0$) the CI equation for the vector field reduces exactly to the Maxwell equation [7] and CI scalar field in this limit leads to the standard CI wave equation in Minkowskian space. For symmetric tensor field of rank-2, the CI system (2.16) for $n = 1$ leads to [2] (for simplicity from now on we take H=1 ):

$$(Q_0 - 2)\mathcal{K}_{\alpha\beta} + \frac{2}{3}\mathcal{S}(\bar{\partial}_\beta + 2x_\beta)\bar{\partial}\cdot\mathcal{K}_\alpha - \frac{1}{3}\theta_{\alpha\beta}\bar{\partial}\cdot\bar{\partial}\cdot\mathcal{K} = 0\,, \tag{3.1}$$

By imposing the traceless and divergenceless conditions on the tensor field $\mathcal{K}_{\alpha\beta}$, which are necessary for UIRs of dS group, the CI equation (3.1) reduces to

$$(Q_0 - 2)\mathcal{K}_{\alpha\beta} = 0, \text{ or equivalently } (Q_2 + 4)\mathcal{K}_{\alpha\beta} = 0\ , \tag{3.2}$$

we name this CI field equation an auxiliary field equation. The relation (2.12) indicates that the solution of this field equation coincides with the discrete series normally $\Pi^{\pm}_{2,q=1}$ [20]. Note that in the flat limit the CI equation (3.1) reduces to CI massless spin-2 wave equation of order-2 in four dimensional Minkowski space which was found by Barut and Xu [2]. They have obtained this equation by varying the coefficients of various term in the standard equation [21].

Now we want to obtain the solution for this auxiliary field. We write this solution in general form as [7, 22]

$$\mathcal{K}_{\alpha\beta} = \theta_{\alpha\beta}\phi_1 + \mathcal{S}\bar{Z}_{1\alpha}K_\beta + D_{2\alpha}K_{g\beta}, \tag{3.3}$$

where the operator $D_2$ is the generalized gradient defined by

$$D_{2\alpha}K_\beta = \mathcal{S}(\bar{\partial}_\alpha - x_\alpha)K_\beta,$$

and $Z_1$ is a constant 5-dimensional vector, $\phi_1$ is a scalar field, $K$ and $K_g$ are two vector fields. The divergenceless and transversality conditions together with $\mathcal{K}' = 0$ result in

$$x\cdot K = 0 = x\cdot K_g \text{ and } 2\phi_1 + Z_1.K + \bar{\partial}.K_g = 0. \tag{3.4}$$

By substituting $\mathcal{K}_{\alpha\beta}$ in (3.2) we obtain

$$\left\{ \begin{array}{rcll} (Q_0+4)\phi_1 &=& -4Z_1.K, & (I) \\ Q_1 K_\beta &=& 0, & (II) \\ (Q_1+4)K_{g\beta} &=& 2(x.Z_1)K_\beta. & (III) \end{array} \right. \tag{3.5}$$

Using conditions $x.K = 0 = \bar{\partial}.K$, Eq.$(3.5-II)$ reduces to $(Q_0-2)K_\beta = 0$. From this reduced form and Eq.$(3.5-I)$, we can write

$$\phi_1 = -\frac{2}{3}Z_1.K, \quad \text{which satisfies} \quad (Q_0-2)\phi_1 = 0, \tag{3.6}$$

and from Eq.(3.4), we have

$$\bar{\partial}.K_g = \frac{1}{3}Z_1.K. \tag{3.7}$$

In continue to our solution, similar to (3.3) we can choose the following form for the vector field $K$ (the solution of $(3.5-II)$) [5, 23]

$$K_\alpha = \bar{Z}_{2\alpha}\phi_2 + D_{1\alpha}\phi_3, \tag{3.8}$$

where $D_1 = \bar{\partial}$ and $Z_2$ is another 5-dimensional constant vector, $\phi_2$ and $\phi_3$ are two scalar fields which will be identified later. Substituting $K$ into $(3.5-II)$ results in

$$(Q_0-2)\phi_2 = 0, \tag{3.9}$$

it is clear that $\phi_2$ is a massless conformally coupled scalar field. $\phi_3$ can be written in terms of $\phi_2$ (appendix B)

$$\phi_3 = (\ x.Z_2)\phi_2. \tag{3.10}$$

So we can write

$$K_\alpha = \left(\bar{Z}_{2\alpha} + \ D_{1\alpha}(x.Z_2)\right)\phi_2, \tag{3.11}$$

and

$$\phi_1 = -\frac{2}{3}Z_1.\left(\bar{Z}_2 + \ D_1(x.Z_2)\right)\phi_2. \tag{3.12}$$

According to the following identity (appendix B)

$$(x.Z_1)K_\alpha = \frac{1}{6}(Q_1+4)\left[(x.Z_1)K_\alpha + \frac{1}{9}D_{1\alpha}(Z_1.K)\right], \tag{3.13}$$

Eq.$(3.5-III)$ leads to

$$K_{g\alpha} = \frac{1}{3}\left[(x.Z_1)K_\alpha + \frac{1}{9}D_{1\alpha}(Z_1.K)\right] + \Lambda_\alpha, \tag{3.14}$$

where $x.K_g = 0$ and $\bar{\partial}.K_g = \frac{1}{3}Z_1.K$ and $\Lambda$ is a gauge field with the following conditions

$$(Q_1+4)\Lambda_\alpha = 0, \text{ with } x\cdot\Lambda = 0\,,\ \bar{\partial}\cdot\Lambda = 0. \tag{3.15}$$

Finally using the Eq.s (3.11), (3.12) and (3.14), we can rewrite $\mathcal{K}_{\alpha\beta}$ in the following form

$$\mathcal{K}_{\alpha\beta}(x) = \mathcal{D}_{\alpha\beta}(x, \partial, Z_1, Z_2)\phi_2, \tag{3.16}$$

where $\mathcal{D}$ is the projector tensor

$$\mathcal{D}(x,\partial,Z_1,Z_2)=\left(-\frac{2}{3}\theta Z_1.+\mathcal{S}\bar{Z}_1+\frac{1}{3}D_2\left[\frac{1}{9}D_1Z_1.+x.Z_1\right]\right)$$
$$\left(\bar{Z}_2+\ D_1\ (x.Z_2)\right). \tag{3.17}$$

# 4 Two-point function

The Wightman two-point function $\mathcal{W}$ is defined by

$$\mathcal{W}_{\alpha\beta\alpha'\beta'}(x,x')=\langle\Omega|\mathcal{K}_{\alpha\beta}(x)\mathcal{K}_{\alpha'\beta'}(x')|\Omega\rangle, \tag{4.1}$$

where $x,x'\in X_H$ and $|\Omega\rangle$ is the Fock vacuum state. This function which is a solution of the wave Eq.(3.2) with respect to $x$ or $x'$, can be found simply in terms of the scalar two-point function. It will be shown that this two-point function can be written as follows

$$\mathcal{W}_{\alpha\beta\alpha'\beta'}(x,x')=\Delta_{\alpha\beta\alpha'\beta'}\mathcal{W}_c(x,x'),$$

where $\mathcal{W}_c(x,x')$ is the scalar two-point function and $\Delta_{\alpha\beta\alpha'\beta'}$ is the bi-tensor projection operator. We consider the following possibility for the transverse two-point function

$$\mathcal{W}(x,x')=\theta\theta'\mathcal{W}_0(x,x')+\mathcal{S}\mathcal{S}'\theta.\theta'\mathcal{W}_1(x,x')+D_2D_2'\mathcal{W}_g(x,x'), \tag{4.2}$$

where $D_2D_2'=D_2'D_2$ and $\mathcal{W}_1$ and $\mathcal{W}_g$ are transverse bi-vector two-point functions. At this stage it is shown that calculation of $\mathcal{W}(x,x')$ could be initiated from either $x$ or $x'$ without any difference that means each choices result in the same equation for $\mathcal{W}(x,x')$. We first consider the choice $x$. In this case $\mathcal{W}(x,x')$ must satisfy the Eq.(3.2), therefor it is easy to show that:

$$\left\{\begin{array}{rcll}(Q_0+4)\theta'\mathcal{W}_0&=&-4\mathcal{S}'\theta'.\mathcal{W}_1, & (I)\\ Q_1\mathcal{W}_1&=&0, & (II)\\ (Q_1+4)D_2'\mathcal{W}_g&=&2\mathcal{S}'(x.\theta')\mathcal{W}_1. & (III)\end{array}\right. \tag{4.3}$$

Using the condition $\partial.\mathcal{W}_1=0$, Eq.$(4.3-I)$ leads to

$$\theta'\mathcal{W}_0(x,x')=-\frac{2}{3}\mathcal{S}'\theta'.\mathcal{W}_1(x,x'). \tag{4.4}$$

The solution of the Eq.$(4.3-II)$ can be written as the combination of two arbitrary bi-scalar two-point functions $\mathcal{W}_2$ and $\mathcal{W}_3$ in the following form

$$\mathcal{W}_1=\theta.\theta'\mathcal{W}_2+D_1D_1'\mathcal{W}_3.$$

Substituting this in Eq.$(4.3-II)$ and using the divergenceless condition we obtain

$$D_1'\mathcal{W}_3=\ x.\theta'\mathcal{W}_2,\ \text{ and }\ (Q_0-2)\mathcal{W}_2=0.$$

This means that $\mathcal{W}_2$ is the massless conformally coupled two-point function. Putting $\mathcal{W}_2 \equiv \mathcal{W}_c$, we have

$$\mathcal{W}_1(x,x') = (\theta.\theta' + \ D_1(\ x.\theta'))\, \mathcal{W}_c(x,x'). \tag{4.5}$$

By using the following identity

$$(Q_0+4)^{-1}(x.\theta')\mathcal{W}_1 = \frac{1}{6}\left[(x.\theta')\mathcal{W}_1 + \frac{1}{9}D_1(\theta'.\mathcal{W}_1)\right],$$

the Eq.$(4.3 - III)$ leads to

$$D'_2\mathcal{W}_g(x,x') = \frac{1}{3}\mathcal{S}'\left(\frac{1}{9}D_1\theta'. + x.\theta'\right)\mathcal{W}_1(x,x'). \tag{4.6}$$

According to Eq.s (4.4), (4.5) and (4.6) it turns out that the two-point function can be rewritten in the following form

$$\mathcal{W}_{\alpha\beta\alpha'\beta'}(x,x') = \Delta_{\alpha\beta\alpha'\beta'}(x,\partial,x',\partial')\mathcal{W}_c(x,x'), \tag{4.7}$$

where

$$\begin{aligned}\Delta_{\alpha\beta\alpha'\beta'}(x,\partial,x',\partial') = & -\frac{2}{3}\mathcal{S}'\theta\theta'.\left(\theta.\theta' + \ D_1(\theta'.x)\right)\\ & +\mathcal{S}\mathcal{S}'\theta.\theta'\left(\theta.\theta' + \ D_1(\theta'.x)\right)\\ & +\frac{1}{3}D_2\mathcal{S}'\left(\frac{1}{9}D_1\theta'. + x.\theta'\right)\left(\theta.\theta' + \ D_1(\theta'.x)\right).\end{aligned} \tag{4.8}$$

On the other hand with the choice $x'$, the two-point function (4.2) satisfies Eq.(3.2) (with respect to $x'$), and similarly we obtain:

$$\left\{\begin{array}{rcll}(Q'_0+4)\theta\mathcal{W}_0 & = & -4\mathcal{S}\theta.\mathcal{W}_1, & (I)\\ Q'_1\mathcal{W}_1 & = & 0, & (II)\\ (Q'_1+4)D_2\mathcal{W}_g & = & 2\mathcal{S}(x'.\theta)\mathcal{W}_1. & (III)\end{array}\right.$$

Using the condition $\partial'.\mathcal{W}_1 = 0$, we have

$$\theta\mathcal{W}_0(x,x') = -\frac{2}{3}\mathcal{S}\theta.\mathcal{W}_1(x,x'),$$

$$D_2\mathcal{W}_g(x,x') = \frac{1}{3}\mathcal{S}\left(\frac{1}{9}D'_1\theta. + x'.\theta\right)\mathcal{W}_1(x,x'),$$

$$\mathcal{W}_1(x,x') = (\theta'.\theta + \ D'_1(\ x'.\theta))\,\mathcal{W}_c(x,x'),$$

where the primed operators act on the primed coordinates only. In this case, the two-point function can be written in the following form

$$\mathcal{W}_{\alpha\beta\alpha'\beta'}(x,x') = \Delta'_{\alpha\beta\alpha'\beta'}(x,\partial,x',\partial')\mathcal{W}_c(x,x'),$$

where

$$\begin{aligned}\Delta'_{\alpha\beta\alpha'\beta'}(x,\partial,x',\partial') = & -\frac{2}{3}\mathcal{S}\theta\theta'.\left(\theta.\theta' + \ D'_1(\theta.x')\right)\\ & +\mathcal{S}\mathcal{S}'\theta.\theta'\left(\theta.\theta' + \ D'_1(\theta.x')\right)\end{aligned}$$

$$+\frac{1}{3}D'_2\mathcal{S}\left(\frac{1}{9}D'_1\theta. + x'.\theta\right)(\theta.\theta' + \ D'_1(\theta.x'))\,.$$

In a few steps ahead, it is shown that this equation is non other than Eq.(4.8).

The conformally coupled scalar field two-point function is [24]:

$$\mathcal{W}_c(x,x') = -\frac{1}{8\pi^2}\left[\mathcal{P}\frac{1}{1-\mathcal{Z}(x,x')} - i\pi\epsilon(x^0-x'^0)\delta(1-\mathcal{Z}(x,x'))\right], \tag{4.9}$$

where $\mathcal{P}$ denotes principal part and the geodesic distance is implicitly defined for $\mathcal{Z} = -x\cdot x'$, by: 1) $\mathcal{Z} = \cosh(\sigma)$ if $x$ and $x'$ are time-like separated, 2) $\mathcal{Z} = \cos(\sigma)$ if $x$ and $x'$are space-like separated where $\sigma$ is the distance along the geodesic connecting the points $x$ and $x'$ (note that $\sigma(x,x')$ can be defined by an unique analytic extension also when no geodesic connects $x$ and $x'$), and also

$$\epsilon(x^0-x'^0) = \left\{\begin{array}{rl} 1 & x^0 > x'^0, \\ 0 & x^0 = x'^0, \\ -1 & x^0 < x'^0. \end{array}\right. \tag{4.10}$$

Eq.s (4.4), (4.5), (4.6) and (4.9) after relatively simple and straightforward calculations can be written as (appendix A):

$$\theta'_{\alpha'\beta'}\mathcal{W}_0(x,x') = -\frac{2}{3}\mathcal{S}'\left[2\theta'_{\alpha'\beta'} + (x.\theta'_{\alpha'})(x.\theta'_{\beta'})(2+\mathcal{Z}\frac{d}{d\mathcal{Z}})\right]\mathcal{W}_c(\mathcal{Z}), \tag{4.11}$$

$$\mathcal{W}_{1\beta\beta'}(x,x') = \left[-(x'.\theta_\beta)(x.\theta'_{\beta'})\frac{d}{d\mathcal{Z}} + 2(\theta_\beta.\theta'_{\beta'})\right]\mathcal{W}_c(\mathcal{Z}), \tag{4.12}$$

$$\begin{aligned} D_{2\alpha}D'_{2\alpha'}\mathcal{W}_{g\beta\beta'}(x,x') = {} & \frac{1}{27(1-\mathcal{Z}^2)^2}\mathcal{S}\mathcal{S}'\Big[\theta_{\alpha\beta}\theta'_{\alpha'\beta'}(1-\mathcal{Z}^2)^2(2\mathcal{Z}\frac{d}{d\mathcal{Z}}) \\ & +(x'.\theta_\alpha)(x'.\theta_\beta)(x.\theta'_{\alpha'})(x.\theta'_{\beta'})((26-14\mathcal{Z}^2)+(58\mathcal{Z}-34\mathcal{Z}^3)\frac{d}{d\mathcal{Z}}) \\ & +\theta'_{\alpha'\beta'}(x'.\theta_\alpha)(x'.\theta_\beta)(1-\mathcal{Z}^2)(4+8\mathcal{Z}\frac{d}{d\mathcal{Z}}) \\ & +\theta_{\alpha\beta}(x.\theta'_{\alpha'})(x.\theta'_{\beta'})(1-\mathcal{Z}^2)((22-20\mathcal{Z}^2)+(14\mathcal{Z}-10\mathcal{Z}^3)\frac{d}{d\mathcal{Z}}) \\ & +(\theta_\alpha.\theta'_{\alpha'})(\theta_\beta.\theta'_{\beta'})(1-\mathcal{Z}^2)^2(22+2\mathcal{Z}\frac{d}{d\mathcal{Z}}) \\ & -(\theta_\alpha.\theta'_{\alpha'})(x.\theta'_{\beta'})(x'.\theta_\beta)(1-\mathcal{Z}^2)(8\mathcal{Z}+(48-32\mathcal{Z}^2)\frac{d}{d\mathcal{Z}})\Big]\mathcal{W}_c(\mathcal{Z}), \end{aligned} \tag{4.13}$$

where

$$(Q_0-2)\mathcal{W}_c(\mathcal{Z}) = 0.$$

Now we are in a position to write the final form of the two-point function in ambient space. Substituting Eq.s(4.11), (4.12) and (4.13) in (4.2) yields

$$\mathcal{W}_{\alpha\beta\alpha'\beta'}(x,x') = \frac{1}{27}\mathcal{S}\mathcal{S}'\left[\theta_{\alpha\beta}\theta'_{\alpha'\beta'}f_1(\mathcal{Z}) + (\theta_\alpha.\theta'_{\alpha'})(\theta_\beta.\theta'_{\beta'})f_2(\mathcal{Z})\right.$$

$$+\theta'_{\alpha'\beta'}(x'.\theta_\alpha)(x'.\theta_\beta)f_3(\mathcal{Z})+(x'.\theta_\alpha)(x'.\theta_\beta)(x.\theta'_{\alpha'})(x.\theta'_{\beta'})f_4(\mathcal{Z})+(\theta_\alpha.\theta'_{\alpha'})(x.\theta'_{\beta'})(x'.\theta_\beta)f_5(\mathcal{Z})$$

$$+\theta_{\alpha\beta}(x.\theta'_{\alpha'})(x.\theta'_{\beta'})f_6(\mathcal{Z})\Big]\,\mathcal{W}_c(\mathcal{Z}). \tag{4.14}$$

where

$$f_1(\mathcal{Z})=(-18+2\mathcal{Z}\frac{d}{d\mathcal{Z}}),\ \ f_2(\mathcal{Z})=(76+2\mathcal{Z}\frac{d}{d\mathcal{Z}}),$$

$$f_3(\mathcal{Z})=(1-\mathcal{Z}^2)^{-1}\left(4+8\mathcal{Z}\frac{d}{d\mathcal{Z}}\right),$$

$$f_4(\mathcal{Z})=(1-\mathcal{Z}^2)^{-2}\left((26-14\mathcal{Z}^2)+(58\mathcal{Z}-34\mathcal{Z}^3)\frac{d}{d\mathcal{Z}}\right),$$

$$f_5(\mathcal{Z})=-(1-\mathcal{Z}^2)^{-1}\left(8\mathcal{Z}+(75-59\mathcal{Z}^2)\frac{d}{d\mathcal{Z}}\right),$$

$$f_6(\mathcal{Z})=(1-\mathcal{Z}^2)^{-1}\left((4-2\mathcal{Z}^2)+(5\mathcal{Z}-\mathcal{Z}^3)\frac{d}{d\mathcal{Z}}\right).$$

Eq.(4.14) is the explicit form of the two-point function in 5 dimensional flat space (ambient space). This equation satisfies the traceless and divergenceless conditions:

$$\bar{\partial}.\mathcal{W}=\bar{\partial}'.\mathcal{W}=0,\quad \text{and}\quad \mathcal{W}_{\alpha\beta\alpha'}{}^{\alpha'}(x,x')=\mathcal{W}^{\alpha}{}_{\alpha\alpha'\beta'}(x,x')=0.$$

The two-point function (4.14) is obviously dS-invariant.
Now it is straightforward to translate this two point function on 4 dimensional de Sitter hyperboloid (intrinsic coordinate)as follows (appendix C)

$$Q_{\mu\nu\mu'\nu'}(X,X')=\frac{1}{27}\mathcal{S}\mathcal{S}'\left[\,g_{\mu\nu}g'_{\mu'\nu'}\,\frac{f_1}{(1-\mathcal{Z}^2)^2}+g_{\mu\mu'}g_{\nu\nu'}\,\frac{f_2}{(1-\mathcal{Z}^2)^2}\right.$$

$$+\,g'_{\mu'\nu'}n_\mu n_\nu\,\frac{f_3}{1-\mathcal{Z}^2}+g_{\mu\mu'}n_\nu n_{\nu'}\left(\frac{2(\mathcal{Z}-1)f_2}{(1-\mathcal{Z}^2)^2}+\frac{f_5}{1-\mathcal{Z}^2}\right)$$

$$\left.+n_\mu n_\nu n_{\mu'}n_{\nu'}\left(\frac{f_2}{(1+\mathcal{Z})^2}-\frac{f_5}{1+\mathcal{Z}}+f_4\right)+g_{\mu\nu}n_{\mu'}n_{\nu'}\,\frac{f_6}{1-\mathcal{Z}^2}\right]\mathcal{W}_c(\mathcal{Z}). \tag{4.15}$$

The two-point function (4.15) is clearly dS-invariant.

# 5 Conclusion

Conformal invariance is the important common property for all equations of massless fields; for example massless vector field (photon) satisfies Maxwell equation which is CI. So, it seems to be important to obtain a CI equation for the massless spin-2 field. Einstein's equation in its linear form is often interpreted as the equation for spin-2 massless field (graviton) in a fixed background metric. Einstein's classical theory of gravitation is not CI, thus could not be considered as a comprehensive universal theory of gravitational fields.
In the Dirac's six-cone formalism by setting $n=1$ in (2.16) we obtained the CI wave equation for

$\mathcal{K}_{\alpha\beta}$ which transformed according to the UIR ($\Pi^{\pm}_{2,1}$) in dS space [2]. Minkowskian limit ($H \rightarrow 0$) of this CI equation coincides to what reported in Ref. [21]. In this paper we considered the solution of this equation. Imposing the conditions divergencelessness and transversality on this field we obtained $(Q_2 + 4)\mathcal{K}_{\alpha\beta} = 0$. On the other hand it has been shown that rank-2 tensor field which is the UIR of the dS and conformal groups satisfies following equation [25]

$$(Q_2 + 4)^2 (Q_2 + 6)\mathcal{K}_{\alpha\beta} = 0,$$

this 6-order differential equation in its non-linear form leads to $R^3$-gravity in dS space. The solutions of the second part of the above equation ($(Q_2 + 6)\mathcal{K}_{\alpha\beta} = 0$) has been considered previously. In this paper we studied the solutions of the $(Q_2+4)\mathcal{K}_{\alpha\beta} = 0$. We find the solution for this CI field equation in terms of the massless conformally coupled scalar field. Related two-point function was also written in terms of the massless conformally coupled two-point function. These two-point function is then dS-invariant. This method may pay the road in considering the field equation of the higher order derivative theories in dS space especially $R^3$ theory of gravity.

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# A Some useful relations

In this appendix, some useful relations are given:

$$Q_1 D_1 = D_1 Q_0, \tag{A.1}$$

$$(Q_0 - 2)x_\alpha = x_\alpha Q_0 - 6x_\alpha - 2\bar{\partial}_\alpha, \tag{A.2}$$

$$\bar{\partial}_\alpha (Q_0 - 2) = Q_0 \bar{\partial}_\alpha - 8\bar{\partial}_\alpha - 2Q_0 x_\alpha - 8x_\alpha, \tag{A.3}$$

$$x_\alpha Q_0 (Q_0 - 2) = (Q_0 - 2)(Q_0 x_\alpha + 4x_\alpha + 4\bar{\partial}_\alpha), \tag{A.4}$$

$$[Q_0 Q_2, Q_2 Q_0]\mathcal{K}_{\alpha\beta} = 4\mathcal{S}(x_\alpha - \bar{\partial}_\alpha)\bar{\partial}.\mathcal{K}_\beta. \tag{A.5}$$

Following relations become important in deriving two-point function

$$\bar{\partial}_\alpha f(\mathcal{Z}) = -(x'.\theta_\alpha)\frac{df(\mathcal{Z})}{d\mathcal{Z}}, \tag{A.6}$$

$$\theta^{\alpha\beta}\theta'_{\alpha\beta} = \theta..\theta' = 3 + \mathcal{Z}^2, \tag{A.7}$$

$$(x.\theta'_{\alpha'})(x.\theta'^{\alpha'}) = \mathcal{Z}^2 - 1, \tag{A.8}$$

$$(x.\theta'_\alpha)(x'.\theta^\alpha) = \mathcal{Z}(1 - \mathcal{Z}^2), \tag{A.9}$$

$$\bar{\partial}_\alpha (x.\theta'_{\beta'}) = \theta_\alpha.\theta'_{\beta'}, \tag{A.10}$$

$$\bar{\partial}_\alpha (x'.\theta_\beta) = x_\beta (x'.\theta_\alpha) - \mathcal{Z}\theta_{\alpha\beta}, \tag{A.11}$$

$$\bar{\partial}_\alpha (\theta_\beta.\theta'_{\beta'}) = x_\beta(\theta_\alpha.\theta'_{\beta'}) + \theta_{\alpha\beta}(x.\theta'_{\beta'}), \tag{A.12}$$

$$\theta'^{\beta}_{\alpha'}(x'.\theta_\beta) = -\mathcal{Z}(x.\theta'_{\alpha'}), \tag{A.13}$$

$$\theta'^{\gamma}_{\alpha'}(\theta_\gamma.\theta'_{\beta'}) = \theta'_{\alpha'\beta'} + (x.\theta'_{\alpha'})(x.\theta'_{\beta'}), \tag{A.14}$$

$$Q_0 f(\mathcal{Z}) = (1 - \mathcal{Z}^2)\frac{d^2 f(\mathcal{Z})}{d\mathcal{Z}^2} - 4\mathcal{Z}\frac{df(\mathcal{Z})}{d\mathcal{Z}}. \tag{A.15}$$

## B Some details on equations (3.10) and (3.13)

Substituting K into (3.5-II) results in $Q_0\phi_3 = 2x.\mathcal{Z}_2\phi_2$. Imposing the divergenceless condition, we get $2x.\mathcal{Z}_2\phi_2 = -\mathcal{Z}_2.\bar{\partial}\phi_2$. Using the above relations and (3.9) and (A.2) one can obtain (3.10).

By using (2.13) it is easy to show that

$$D_1(Z_1.K) = \frac{1}{6}(Q_1+4)[D_1(Z_1.K)], \tag{B.1}$$

$$x(Z_1.K) = \frac{1}{6}(Q_1+4)[x(Z_1.K)], \tag{B.2}$$

$$Z_1.\bar{\partial}K = \frac{1}{6}(Q_1+4)[Z_1.\bar{\partial}K - \frac{1}{3}D_1(Z_1.K)], \tag{B.3}$$

$$(Q_1+4)[(x.Z_1)K] = 2[x(Z_1.K) - Z_1.\bar{\partial}K]. \tag{B.4}$$

The conditions $x.K = \bar{\partial}.K = 0$, and $(Q_0-2)K = 0$, are used to obtain the above equations.

Substituting Eq.s $(B.2)$ and $(B.3)$ in $(B.4)$ we have

$$(Q_1+4)[(x.Z_1)K] = \frac{1}{3}(Q_1+4)\left[\frac{1}{3}D_1(Z_1.K) + x(Z_1.K) - Z_1.\bar{\partial}K\right], \tag{B.5}$$

or

$$(x.Z_1)K = \frac{1}{3}\left[\frac{1}{3}D_1(Z_1.K) + x(Z_1.K) - Z_1.\bar{\partial}K\right]. \tag{B.6}$$

Finally according Eq.s $(B.1)$ and $(B.4)$, we obtain

$$(x.Z_1)K = \frac{1}{6}(Q_1+4)\left[\frac{1}{9}D_1(Z_1.K) + (x.Z_1)K\right]. \tag{B.7}$$

This automatically leads to Eq.(3.13).

## C Relation between the ambient space notation and the intrinsic coordinates

In order to compare our results with the work of the other authors [9, 26], the relation between the ambient space notation and the intrinsic coordinates is studied in the final stage. In order to translate the relations into the ambient coordinates, we use the fact that the "intrinsic" field $h_{\mu\nu}(X)$ is locally determined by the "transverse" tensor field $\mathcal{K}_{\alpha\beta}(x)$ through

$$h_{\mu\nu}(X) = \frac{\partial x^\alpha}{\partial X^\mu}\frac{\partial x^\beta}{\partial X^\nu}\mathcal{K}_{\alpha\beta}(x(X)). \tag{C.1}$$

In the same way one can show that the transverse projector $\theta$ is the only symmetric and transverse tensor which is linked to the dS metric $g_{\mu\nu}$:

$$g_{\mu\nu} = \frac{\partial x^\alpha}{\partial X^\mu}\frac{\partial x^\beta}{\partial X^\nu}\,\theta_{\alpha\beta}.$$

Covariant derivatives acting on a symmetric, second rank tensor are transformed according to

$$\nabla_\rho \nabla_\lambda h_{\mu\nu} = \frac{\partial x^\alpha}{\partial X^\rho}\frac{\partial x^\beta}{\partial X^\lambda}\frac{\partial x^\gamma}{\partial X^\mu}\frac{\partial x^\delta}{\partial X^\nu}\mathrm{Trpr}\bar\partial_\alpha \mathrm{Trpr}\bar\partial_\beta \mathcal{K}_{\gamma\delta}. \tag{C.2}$$

The transverse projection (Trpr) defined by

$$(\mathrm{Trpr}\mathcal{K})_{\alpha\beta} = \theta_\alpha^\gamma \theta_\beta^\delta \mathcal{K}_{\gamma\delta},$$

guarantees the transversality in each index. For example we have [7]

$$\nabla_\rho \nabla_\lambda h_{\mu\nu} = \frac{\partial x^\alpha}{\partial X^\rho}\frac{\partial x^\beta}{\partial X^\lambda}\frac{\partial x^\gamma}{\partial X^\mu}\frac{\partial x^\delta}{\partial X^\nu}[\bar\partial_\alpha(\bar\partial_\beta \mathcal{K}_{\gamma\delta} - x_\gamma \mathcal{K}_{\beta\delta} - x_\delta \mathcal{K}_{\beta\gamma})$$
$$-x_\beta(\bar\partial_\alpha \mathcal{K}_{\gamma\delta} - x_\gamma \mathcal{K}_{\alpha\delta} - x_\delta \mathcal{K}_{\alpha\gamma}) - x_\gamma(\bar\partial_\beta \mathcal{K}_{\alpha\delta} - x_\alpha \mathcal{K}_{\beta\delta} - x_\delta \mathcal{K}_{\beta\alpha})$$
$$-x_\delta(\bar\partial_\beta \mathcal{K}_{\gamma\alpha} - x_\gamma \mathcal{K}_{\beta\alpha} - x_\alpha \mathcal{K}_{\beta\gamma})]. \tag{C.3}$$

By contraction of the covariant derivatives, *i.e.* $\nabla_\rho\nabla^\rho$, the d'Alambertian operator becomes:

$$\Box_H h_{\mu\nu} = g^{\lambda\rho}\nabla_\lambda\nabla_\rho h_{\mu\nu} = \frac{\partial x^\alpha}{\partial X^\mu}\frac{\partial x^\beta}{\partial X^\nu}([\bar\partial_\gamma\bar\partial^\gamma - 2]\mathcal{K}_{\alpha\beta} - 2\mathcal{S}x_\alpha(\bar\partial.\mathcal{K})_\beta + 2x_\alpha x_\beta \mathcal{K}'), \tag{C.4}$$

other relations can be found by this way.

As mentioned in [7], any maximally symmetric bi-tensor are functions of two points $(x, x')$ and behave like tensors under coordinate transformations at each points and they can be expressed as a sum of products of three basic tensors. The coefficients in this expansion are functions of the geodesic distance $\sigma(x, x')$. In this sense, these fundamental tensors form a complete set and they can be obtained by differentiating the geodesic distance:

$$n_\mu = \nabla_\mu \sigma(x, x') \quad , \quad n_{\mu'} = \nabla_{\mu'}\sigma(x, x'),$$

and the parallel propagator

$$g_{\mu\nu'} = -c^{-1}(\mathcal{Z})\nabla_\mu n_{\nu'} + n_\mu n_{\nu'}.$$

The basic bi-tensors in ambient space notations are found through

$$\bar\partial_\alpha \sigma(x, x') \quad , \quad \bar\partial'_{\beta'}\sigma(x, x') \quad , \quad \theta_\alpha.\theta'_{\beta'},$$

restricted to the hyperboloid by

$$\mathcal{T}_{\mu\nu'} = \frac{\partial x^\alpha}{\partial X^\mu}\frac{\partial x'^{\beta'}}{\partial X'^{\nu'}}T_{\alpha\beta'}.$$

For $\mathcal{Z} = \cos(\sigma)$, one can find

$$n_\mu = \frac{\partial x^\alpha}{\partial X^\mu}\bar\partial_\alpha \sigma(x, x') = \frac{\partial x^\alpha}{\partial X^\mu}\frac{(x'\cdot\theta_\alpha)}{\sqrt{1-\mathcal{Z}^2}}, \quad n_{\nu'} = \frac{\partial x'^{\beta'}}{\partial X'^{\nu'}}\bar\partial_{\beta'}\sigma(x, x') = \frac{\partial x'^{\beta'}}{\partial X'^{\nu'}}\frac{(x\cdot\theta'_{\beta'})}{\sqrt{1-\mathcal{Z}^2}},$$

$$\nabla_\mu n_{\nu'} = \frac{\partial x^\alpha}{\partial X^\mu}\frac{\partial x'^{\beta'}}{\partial X'^{\nu'}}\theta_\alpha^\varrho \theta'^{\gamma'}_{\beta'}\bar\partial_\varrho\bar\partial_{\gamma'}\sigma(x, x') = c(\mathcal{Z})[n_\mu n_{\nu'}\mathcal{Z} - \frac{\partial x^\alpha}{\partial X^\mu}\frac{\partial x'^{\beta'}}{\partial X^{\nu'}}\theta_\alpha\cdot\theta'_{\beta'}],$$

with $c^{-1}(\mathcal{Z}) = -\frac{1}{\sqrt{1-\mathcal{Z}^2}}$. For $\mathcal{Z} = \cosh(\sigma)$, $n_\mu$ and $n_\nu$ are multiplied by $i$ and $c(\mathcal{Z})$ becomes $-\frac{i}{\sqrt{1-\mathcal{Z}^2}}$. In both cases we have

$$g_{\mu\nu'} + (\mathcal{Z}-1)n_\mu n_{\nu'} = \frac{\partial x^\alpha}{\partial X^\mu}\frac{\partial x'^{\beta'}}{\partial X'^{\nu'}}\theta_\alpha \cdot \theta'_{\beta'}.$$

and the two-point functions are related through

$$Q_{\mu\nu\mu'\nu'} = \frac{\partial x^\alpha}{\partial X^\mu}\frac{\partial x^\beta}{\partial X^\nu}\frac{\partial x'^{\alpha'}}{\partial X'^{\mu'}}\frac{\partial x'^{\beta'}}{\partial X'^{\nu'}}\mathcal{W}_{\alpha\beta\alpha'\beta'}.$$

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